Dynamical solution of the on-line minority game

2001 J. Phys. A: Math. Gen. 34 10783

(http://iopscience.iop.org/0305-4470/34/49/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10783–10804 PII: S0305-4470(01)27460-5

Dynamical solution of the on-line minority game

A C C Coolen and J A F Heimel

Department of Mathematics, King's College London, The Strand, London WC2R 2LS, UK

Received 1 August 2001
Published 30 November 2001
Online at stacks.iop.org/JPhysA/34/10783

Abstract

We solve the dynamics of the on-line minority game (MG), with general types of decision noise, using generating functional techniques a la De Dominicis and the temporal regularization procedure of Bedeaux *et al*. The result is a macroscopic dynamical theory in the form of closed equations for correlation and response functions defined via an effective continuous-time single-trader process, which are exact in both the ergodic and in the non-ergodic regime of the MG. Our solution also explains why, although one cannot formally truncate the Kramers–Moyal expansion of the process after the Fokker–Planck term, upon doing so one still finds the correct solution, that the previously proposed diffusion matrices for the Fokker–Planck term are incomplete, and how previously proposed approximations of the market volatility can be traced back to ergodicity assumptions.

PACS numbers: 02.50.Le, 87.23.Ge, 05.70.Ln, 64.60.Ht

1. Introduction

The minority game (MG) [1] is an intriguing variation on the so-called El-Farol bar problem [2] which aims to capture and understand in the simplest possible way the cooperative phenomena in markets of interacting traders. Its equations describe the stochastic evolution in time of the selection of individual 'trading strategies' by a community of traders which operate in a simple market. The rules of this market are that each trader has to make a binary decision at every point in time (e.g. whether to buy or sell), and that profit is made only by those traders who find themselves in the minority group (i.e. who find themselves buying when most wish to sell, or vice versa). The essence of the MG is that each trader individually wishes to make profit, but that the net effect of his/her trading actions is defined fully in terms of (or relative to) the actions taken by the other traders, and that there is a high degree of frustration since it is impossible for all traders to be successful at the same time. In spite of its apparent simplicity, the MG has been found to exhibit non-trivial behaviour (e.g. phase transitions separating an ergodic from a highly non-ergodic regime) and to pose a considerable challenge to the theorist. Its (stochastic) equations do not obey detailed balance, so there is no equilibrium state, and understanding the model properly inevitably requires solving its dynamics. An overview of the literature on the MG and its many variations and extensions can be found in [3].

In the original MG, the information supplied to the agents upon which to base their trading decisions consisted of the history of the market. However, it was realized [4] that the dynamics of the MG remains largely unaltered if, instead of the true history of the market, random information is supplied to the agents; given α (the relative number of possible values for the external information), the only relevant condition is that all agents must be given the *same* information (whether sensible or otherwise). This led to a considerable simplification of theoretical approaches to the MG, since it reduced the process to a Markovian one. In [5] it was shown that the relevant quantities studied in the MG can be scaled such that they become independent of the number of agents N when this number becomes very large, opening up the possibility to use statistical mechanical tools. An interesting generalization of the game was the introduction of agents' decision noise [6], which was shown not only to improve worse than random behaviour but also, more surprisingly, to be able to make it better than random[1]. The studies [6] and [10] finally paved the way for a number of papers aiming to develop a solvable statistical mechanical theory.

Early theoretical attempts focused on using additive decision noise to regularize the microscopic stochastic laws and derive deterministic (nonlinear) continuous-time equations, which minimized a Lyapunov function [11, 12]. This approach was remarkably successful in identifying the location of a phase transition and in describing correctly some aspects of the behaviour of the MG above this phase transition. However, it became clear later that the deterministic equations were only approximate, even for $N \to \infty$ (since they neglected relevant fluctuations). This initiated a debate about how to construct exact continuous-time microscopic laws for the MG [13–17], in which all participants restricted themselves to either deterministic or Fokker–Planck equations (some dealing with both additive and multiplicative decision noise [15]) but without agreeing on the expression to be used for the diffusion matrix, and all involving different types of approximations already at the microscopic level. Finally, in [20, 21] the problems relating to the continuous-time limit were circumvented by redefining the equations at the microscopic level directly in terms of full averages over all possible values of the external information, and the dynamics of the resulting so-called 'batch MG' was solved exactly using generating functional methods [19] (first without decision noise [20], then also for general types of decision noise [21]).

In this paper we solve the dynamics of the original (on-line) MG, along the lines of [20, 21] (i.e. using generating functional techniques). We show how the problems and uncertainties relating to temporal regularization, which appear to have been responsible for generating the debates and approximations surrounding the proper form of the continuous-time microscopic laws, can be solved and resolved in an elegant and transparent way by using the (exact) procedure of [18] for deriving a continuous-time master equation without using the thermodynamic limit. This allows us to write down the full stochastic microscopic equations (without truncation or approximation) and solve the model by adapting to the present continuous-time (on-line) process the generating functional techniques which were employed successfully for the discrete-time (batch) process in [20, 21]. The end result is an exact dynamical macroscopic theory, for general types of decision noise (additive, multiplicative, etc), in the form of closed equations for correlation and response functions which are defined via an effective single-trader process, from which the statics follows as a spin-off. Our equations describe both the ergodic and the non-ergodic regime of the MG, including transients.

[1] Using a phenomenological theory for the volatility, based on so-called 'crowd–anticrowd' cancellations [7], this effect was partially explained in [8, 9].

The full exact dynamical solution now available also allows us to make rigorous retrospective statements about the validity or otherwise of the various approximations proposed in the past, and to explain under which conditions such approximations could indeed have led to correct results. More specifically, we (i) show why it is in principle not allowed to truncate the Kramers–Moyal expansion of the microscopic process after the Fokker–Planck term (let alone after the flow term), but why upon doing so one can still find the correct macroscopic equations (for the present version of the MG), (ii) confirm that the different diffusion matrices for the Fokker–Planck term in the process, as proposed earlier by others, are incomplete or approximate, and (iii) indicate how previously proposed approximations involving the market volatility can be traced back to assumptions relating to ergodicity (and are thus valid at most in the ergodic regime).

2. Definitions

2.1. The minority game

The MG describes the dynamics of decision making by N interacting trading agents, labelled with Roman indices. At each round ℓ of the game, each agent i has to take a binary trading decision $s_i(\ell) \in \{-1, 1\}$ (e.g. whether to sell or buy). At each round all agents are given the same external information $I_{\mu(\ell)}$ (representing, for example, the overall state of the market, political or economic developments, etc), which here is chosen randomly and independently from a total number $p = \alpha N$ of possible values, i.e. $\mu(\ell) \in \{1, \ldots, \alpha N\}$ for each ℓ. To generate trading decisions, each agent i has S fixed strategies $\boldsymbol{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^{\alpha N}) \in \{-1, 1\}^{\alpha N}$ at his/her disposal, with $a \in \{1, \ldots, S\}$. These strategies act as manuals (or look-up tables) for decision making: if strategy a is being used by agent i at stage ℓ in the game, then the observation of external information $\mu(\ell)$ will trigger this particular agent into taking the trading decision $s_i(\ell) = R_{ia}^{\mu(\ell)}$. Hence the intrinsic dynamics of the MG is not driven by the decision variables $s_i(\ell)$, but by the dynamic selection by each trader of trading strategies from his/her available arsenal, as described below. Each component R_{ia}^{μ} is assumed to have been drawn randomly and independently from $\{-1, 1\}$ before the start of the game, with uniform probabilities. The strategies thus introduce quenched disorder into the game.

Given a choice $\mu(\ell)$ made at the start of round ℓ, every agent i selects the strategy $\tilde{a}_i(\ell)$ which for trader i has the highest pay-off value at that point in time, i.e. $\tilde{a}_i(\ell) = \arg\max\, p_{ia}(\ell)$, and subsequently makes the binary bid $b_i(\ell) = R_{i\tilde{a}_i(\ell)}^{\mu(\ell)}$. The (re-scaled) total bid at stage ℓ is defined as $A(\ell) = N^{-1/2} \sum_i b_i(\ell)$. Next all agents update the pay-off values of *each* of their strategies a on the basis of what would have happened if they had played that particular strategy:

$$p_{ia}(\ell + 1) = p_{ia}(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} R_{ia}^{\mu(\ell)} A(\ell). \tag{1}$$

The constant $\tilde{\eta}$ represents an (optional) learning rate. Note that the agents all behave as price takers, i.e. they do not take into account the impact of their own decisions on the total bid. We here consider only the $S = 2$ model, where the equations can be simplified upon introducing $q_i(\ell) = \frac{1}{2}[p_{i1}(\ell) - p_{i2}(\ell)]$, $\boldsymbol{\omega}_i = \frac{1}{2}[\boldsymbol{R}_{i1} + \boldsymbol{R}_{i2}]$ and $\boldsymbol{\xi}_i = \frac{1}{2}[\boldsymbol{R}_{i1} - \boldsymbol{R}_{i2}]$. As a result, the selected strategy in round ℓ can now be written as $\boldsymbol{R}_{i\tilde{a}_i(\ell)} = \boldsymbol{\omega}_i + \mathrm{sgn}\,[q_i(\ell)]\boldsymbol{\xi}_i$, and the evolution of the difference is given by

$$q_i(\ell + 1) = q_i(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} \xi_i^{\mu(\ell)} \left[\Omega^{\mu(\ell)} + \frac{1}{\sqrt{N}} \sum_j \xi_j^{\mu(\ell)} \mathrm{sgn}\,[q_j(\ell)] \right] \tag{2}$$

with $\boldsymbol{\Omega} = N^{-1/2} \sum_j \boldsymbol{\omega}_j \in \mathbb{R}^{\alpha N}$.

2.2. Minority game with decision noise

The process (2) can and has been generalized in order to include traders' decision noise [6,11]. This can be done in many different ways; here we try to avoid being unnecessarily specific, and choose a general definition where we replace (as in, e.g. [21])

$$\mathrm{sgn}\,[q_j(\ell)] \;\to\; \sigma[q_j(\ell), z_j(\ell)] \tag{3}$$

in which the $z_j(\ell)$ are independent and zero average random numbers, described by some symmetric distribution $P(z)$ which is normalised according to $\int \mathrm{d}z\, P(z) = \int \mathrm{d}z\, P(z)z^2 = 1$. The function $\sigma[q, z]$ is parametrized by a control parameter $T \geqslant 0$ such that $\sigma[q, z] \in \{-1, 1\}$, with $\lim_{T\to 0} \sigma[q, z] = \mathrm{sgn}\,[q]$ and $\lim_{T\to\infty} \int \mathrm{d}z\, P(z)\sigma[q, z] = 0$, so that T can be interpreted as a measure of the degree of stochasticity in the traders' decision making. Typical examples are additive and multiplicative noise definitions such as

$$\text{additive:} \qquad \sigma[q, z] = \mathrm{sgn}\,[q + Tz] \tag{4}$$

$$\text{multiplicative:} \quad \sigma[q, z] = \mathrm{sgn}\,[q]\,\mathrm{sgn}\,[1 + Tz]. \tag{5}$$

In the first case (4) the noise has the potential to be overruled by the so-called 'frozen' agents, which have $q_i(t) \sim \tilde{q}_i t$ for $t \to \infty$ (see [20]). In the second case the decision noise will even retain its effect for 'frozen' agents (if they exist). These definitions represent situations where for $T > 0$ a trader need not always use his/her 'best' strategy; for $T \to 0$ we revert back to the process (2). The impact of the multiplicative noise (5) can be characterized by the monotonic function

$$\lambda(T) = \int \mathrm{d}z\, P(z)\, \mathrm{sgn}\,[1 + Tz] \tag{6}$$

with $\lambda(0) = 1$ and $\lambda(\infty) = 0$. For e.g. a Gaussian $P(z)$ one has $\lambda(T) = \mathrm{erf}\,[1/\sqrt{2}T]$. We now find (2) being replaced by

$$q_i(\ell+1) = q_i(\ell) - \frac{\tilde{\eta}}{\sqrt{N}}\, \xi_i^{\mu(\ell)} A^{\mu(\ell)}[\boldsymbol{q}(\ell), \boldsymbol{z}(\ell)]. \tag{7}$$

In this expression the quantity $A^{\mu}[\boldsymbol{q}, \boldsymbol{z}]$ denotes the bid which would result upon presentation of information μ, given the microscopic state $\boldsymbol{q}$ of the system and given realization $\boldsymbol{z}$ of the decision noise:

$$A^{\mu}[\boldsymbol{q}, \boldsymbol{z}] = \Omega^{\mu} + \frac{1}{\sqrt{N}} \sum_j \xi_j^{\mu} \sigma[q_j, z_j]. \tag{8}$$

3. Microscopic probabilistic description

3.1. Temporal regularization using the procedure of Bedeaux et al

We convert the discrete-time on-line stochastic process (1) into an explicit Markovian description in terms of probability densities. For the present noisy version of the game one finds a microscopic transition probability density operator $W(\boldsymbol{q}|\boldsymbol{q}')$ which involves an average over the random numbers $\{z_i\}$, indicated by $\langle \ldots \rangle_{\boldsymbol{z}}$:

$$p_{\ell+1}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{q}'\, W(\boldsymbol{q}|\boldsymbol{q}') p_{\ell}(\boldsymbol{q}') \tag{9}$$

$$W(\boldsymbol{q}|\boldsymbol{q}') = \frac{1}{p} \sum_{\mu=1}^{p} \left\langle \prod_i \delta\left[q_i - q_i' + \frac{\tilde{\eta}}{\sqrt{N}} \xi_i^{\mu} A^{\mu}[\boldsymbol{q}', \boldsymbol{z}] \right] \right\rangle_{\boldsymbol{z}}. \tag{10}$$

In order to carry out systematically the temporal coarse-graining and transform the dynamics to an appropriately re-scaled continuous time t, we follow the systematic procedure of [18] and define the *duration* of each of the iteration steps to be a continuous random number, the statistics of which are described by the probability $\pi_\ell(t)$ that at time t precisely ℓ updates have been made. Our new process (including the randomness in step duration) is described by

$$p_t(\boldsymbol{q}) = \sum_{\ell\geqslant 0} \pi_\ell(t) p_\ell(\boldsymbol{q}) = \sum_{\ell\geqslant 0} \pi_\ell(t) \int \mathrm{d}\boldsymbol{q}'\, W^\ell(\boldsymbol{q}|\boldsymbol{q}') p_0(\boldsymbol{q}')$$

and time has become continuous. For $\pi_\ell(t)$ we make the Poisson choice $\pi_\ell(t) = \frac{1}{\ell!}(t/\Delta_N)^\ell \mathrm{e}^{-t/\Delta_N}$. From $\langle\ell\rangle_\pi = t/\Delta_N$ and $\langle\ell^2\rangle_\pi = t/\Delta_N + t^2/\Delta_N^2$ it follows that Δ_N is the average duration of an iteration step, and that the relative deviation in ℓ at a given t vanishes for $\Delta_N \to 0$ as $\sqrt{\langle\ell^2\rangle_\pi - \langle\ell\rangle_\pi^2}/\langle\ell\rangle_\pi = \sqrt{\Delta_N/t}$. This introduction of random step durations thus only introduces uncertainty about where we are on the time axis, which will vanish at the end of the calculation, provided we ensure $\lim_{N\to\infty} \Delta_N = 0$. The properties of the Poisson distribution under temporal derivation lead to

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{q}) = \frac{1}{\Delta_N}\left\{ \int \mathrm{d}\boldsymbol{q}'\, W(\boldsymbol{q}|\boldsymbol{q}') p_t(\boldsymbol{q}') - p_t(\boldsymbol{q})\right\}. \tag{11}$$

3.2. Canonical temporal coarse graining

To find the appropriate scaling with N of Δ_N (and investigate the relation with the Fokker–Planck approximations of [15] and [17] in a subsequent section) it is instructive to expand the master equation (11) in powers of the learning rate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{q}) &= \frac{1}{\Delta_N p}\sum_{\mu=1}^{p} \int \mathrm{d}\boldsymbol{q}'\, p_t(\boldsymbol{q}') \left\langle \delta\left(\boldsymbol{q} - \boldsymbol{q}' + \frac{\tilde{\eta}}{\sqrt{N}} \boldsymbol{\xi}^\mu A^\mu[\boldsymbol{q}', \boldsymbol{z}]\right) - \delta(\boldsymbol{q} - \boldsymbol{q}')\right\rangle_{\boldsymbol{z}} \\ &= \sum_{\ell\geqslant 1} [L_\ell\, p_t](\boldsymbol{q})\end{aligned} \tag{12}$$

with

$$\begin{aligned}[L_\ell\, p](\boldsymbol{q}) = \frac{\tilde{\eta}^\ell}{N^{\ell/2}} &\sum_{n_1,\ldots,n_N\geqslant 0} \frac{\delta_{\ell,\sum_i n_i}}{n_1!\ldots n_N!} \frac{\partial^\ell}{\partial q_1^{n_1}\ldots\partial q_N^{n_N}} \\ &\times \left\{ p(\boldsymbol{q}) \left[\frac{1}{\Delta_N p}\sum_{\mu=1}^{p} (\xi_1^\mu)^{n_1}\ldots(\xi_N^\mu)^{n_N} \langle A^\mu[\boldsymbol{q}, \boldsymbol{z}]^\ell\rangle_{\boldsymbol{z}}\right]\right\}.\end{aligned} \tag{13}$$

From the $\ell = 1$ term in this so-called Kramers–Moyal expansion one reads off the canonical scaling for the temporal coarse-graining timescale Δ_N (in order to guarantee a proper $N \to \infty$ limit):

$$[L_1\, p](\boldsymbol{q}) = \frac{\tilde{\eta}}{\alpha\Delta_N N} \sum_i \frac{\partial}{\partial q_i}\left\{ p(\boldsymbol{q})\left[\frac{\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N}\sum_j \boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j \langle\sigma[q_j, z]\rangle_z\right]\right\}.$$

We are automatically led to the choice $\Delta_N = \mathcal{O}(\tilde{\eta}/N)$ (this gives, for $T = 0$ and for $N \to \infty$, as our first term exactly the batch dynamics studied in [20]), and we must require $\lim_{N\to\infty} \tilde{\eta}/N = 0$ in order to guarantee $\lim_{N\to\infty} \Delta_N = 0$. We now put

$$\Delta_N = \tilde{\eta}/2\alpha N \tag{14}$$

(thereby *en passant* absorbing an additional distracting factor 2α, to find simpler equations later) and find the various terms (13) in the Kramers–Moyal expansion (12) reducing to

$$
[L_\ell\, p](\boldsymbol{q}) = \frac{2\tilde{\eta}^{\ell-1}}{N^{\frac{1}{2}(\ell-1)}} \sum_{n_1,\ldots,n_N\geqslant 0} \frac{\delta_{\ell,\sum_i n_i}}{n_1!\ldots n_N!} \frac{\partial^\ell}{\partial q_1^{n_1}\ldots\partial q_N^{n_N}} \\
\times \left\{ p(\boldsymbol{q}) \left[\frac{1}{\sqrt{N}} \sum_{\mu=1}^{p} (\xi_1^\mu)^{n_1} \ldots (\xi_N^\mu)^{n_N} \langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^\ell \rangle_{\boldsymbol{z}} \right] \right\}. \tag{15}
$$

Note that, according to (14), the present canonical definition of the time unit t implies temporal coarse-graining over $\mathcal{O}(1/\Delta_N) = \mathcal{O}(N/\tilde{\eta})$ iteration steps.

3.3. Canonical scaling of the learning rate

The remaining freedom one has is in the choice of the learning rate $\tilde{\eta}$. In order to find the appropriate choice(s) for the scaling with N of $\tilde{\eta}$ we work out the temporal derivatives of the probability density $P_k(q) = \langle \delta[q - q_k] \rangle$ of individual components of the microscopic state vector, where $\langle f(\boldsymbol{q}) \rangle = \int \mathrm{d}\boldsymbol{q}\, p(\boldsymbol{q}) f(\boldsymbol{q})$ (via integration by parts in the various terms of (12)):

$$
\frac{\mathrm{d}}{\mathrm{d}t} P_k(q) = \sum_{\ell\geqslant 1} \frac{2\tilde{\eta}^{\ell-1}}{N^{\ell/2}\ell!} \frac{\partial^\ell}{\partial q^\ell} \left\langle \delta[q-q_k] \sum_{\mu=1}^{p} (\xi_k^\mu)^\ell \langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^\ell \rangle_{\boldsymbol{z}} \right\rangle \\
= \frac{\partial}{\partial q} \left\langle 2\delta[q-q_k] \left[\frac{\boldsymbol{\xi}_k\cdot\boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N} \sum_j \boldsymbol{\xi}_k\cdot\boldsymbol{\xi}_j \langle \sigma[q_j, z_j] \rangle_{\boldsymbol{z}} \right] \right\rangle \\
+ \tilde{\eta} \frac{\partial^2}{\partial q^2} \left\langle \delta[q-q_k] \frac{1}{N} \sum_{\mu=1}^{p} \left\langle \left[\xi_k^\mu A^\mu[\boldsymbol{q},\boldsymbol{z}]\right]^2 \right\rangle_{\boldsymbol{z}} \right\rangle + \sum_{\ell>2} \mathcal{O}(N^{1-\frac{1}{2}\ell}\tilde{\eta}^{\ell-1}). \tag{16}
$$

We see explicitly that the diffusion term in this equation is of order $\mathcal{O}(\tilde{\eta})$. Hence, in order to prevent the system from being overruled by fluctuations we have to choose $\tilde{\eta} = \mathcal{O}(N^0)$ (with the fluctuations in individual components vanishing altogether as soon as $\lim_{N\to\infty} \tilde{\eta} = 0$). Following [6, 11] and subsequent papers we choose $\tilde{\eta}$ to be a constant which is independent of N in the remainder of this study. As a consequence we find that for $N \to \infty$ the single-trader equation (16) reduces to a Fokker–Planck equation, in agreement with [15] and [17] (although, as we will show in a subsequent section, the diffusion matrices proposed in the latter two studies are both approximations). This, however, does not necessarily imply that the underlying N-agent process can also be described by a Fokker–Planck equation.

4. Properties of the Kramers–Moyal expansion

4.1. Relevance of higher orders

With the definitions $\Delta_N = \tilde{\eta}/2p$ and $\tilde{\eta} = \mathcal{O}(1)$ our full microscopic master equation (11) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{q}) = \frac{2p}{\tilde{\eta}} \int \mathrm{d}\boldsymbol{q}'\, p_t(\boldsymbol{q}') \left\{ \frac{1}{p} \sum_{\mu=1}^{p} \left\langle \delta\left[\boldsymbol{q} - \boldsymbol{q}' + \frac{\tilde{\eta}}{\sqrt{N}} \boldsymbol{\xi}^\mu A^\mu[\boldsymbol{q}',\boldsymbol{z}] \right] \right\rangle_{\boldsymbol{z}} - \delta(\boldsymbol{q}'-\boldsymbol{q}) \right\}. \tag{17}
$$

The Kramers–Moyal expansion takes the final form

$$
\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{q}) = \sum_{\ell\geqslant 1} [L_\ell\, p_t](\boldsymbol{q}) \tag{18}
$$

where

$$[L_\ell p](\boldsymbol{q}) = \frac{2\tilde{\eta}^{\ell-1}}{N^{\frac{1}{2}\ell}} \sum_\mu \sum_{n_1,\ldots,n_N \geqslant 0} \frac{\delta_{\ell,\sum_i n_i}}{n_1!\ldots n_N!} (\xi_1^\mu)^{n_1} \ldots (\xi_N^\mu)^{n_N} \frac{\partial^\ell}{\partial q_1^{n_1}\ldots \partial q_N^{n_N}} \left\{p(\boldsymbol{q})\langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^\ell\rangle_{\boldsymbol{z}}\right\} \tag{19}$$

or, equivalently,

$$[L_\ell p](\boldsymbol{q}) = \frac{2\tilde{\eta}^{\ell-1}}{\ell!} \sum_\mu \left[\frac{1}{\sqrt{N}}\sum_i \xi_i^\mu \frac{\partial}{\partial q_i}\right]^\ell \left\{p(\boldsymbol{q})\langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^\ell\rangle_{\boldsymbol{z}}\right\}. \tag{20}$$

Although for $N \to \infty$ the individual components of the state vector $\boldsymbol{q}$ were seen to have Gaussian fluctuations (16), the cumulative effect on the dynamics of the higher-order ($\ell > 2$) terms in the expansion (18) (including cross-correlations) cannot simply be neglected, since also the *number* of components q_i diverges with N. This can be seen more clearly in (20) than in (19). If we work out the evolution of averages of observables $f(\boldsymbol{q})$, via integration by parts, we find that (18), (20) predict

$$\tilde{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\langle f\rangle = 2\sum_{\ell\geqslant 1}\frac{(-\tilde{\eta})^\ell}{\ell!}\sum_\mu \left\langle \langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^\ell\rangle_{\boldsymbol{z}} \left[\frac{1}{\sqrt{N}}\sum_i \xi_i^\mu \frac{\partial}{\partial q_i}\right]^\ell f\right\rangle.$$

For simple mean-field observables such as $f(\boldsymbol{q}) = N^{-1}\sum_i \zeta_i q_i^{2m}$ (with m independent of N) one has $\left[\frac{1}{\sqrt{N}}\sum_i \xi_i^\mu \frac{\partial}{\partial q_i}\right]^\ell f = \mathcal{O}(N^{-\ell/2})$, and hence the $\ell > 2$ terms in the expansion are of vanishing order. In contrast, for observables which are dominated by fluctuations, this need no longer be the case. For instance, in the present system (with its overall reflection symmetry) one has $f(\boldsymbol{q}) = \frac{1}{\sqrt{N}}\sum_i \zeta_i q_i^{2m+1} = \mathcal{O}(N^0)$; now $\left[\frac{1}{\sqrt{N}}\sum_i \xi_i^\mu \frac{\partial}{\partial q_i}\right]^\ell f = \mathcal{O}(N^{-(\ell-1)/2})$, and also the $\ell = 3$ term could contribute to leading order. Moreover, there is no practical need to truncate the expansion, since (as we will show below) within the generating functional formalism it is perfectly natural to derive an analytical solution of the dynamics with all terms of the Kramers–Moyal expansion included.

4.2. *Explicit form of flow and diffusion terms*

Let us now, by way of illustration, work out the first few terms of (20) for additive noise with $P(z) = \frac{1}{2}K[1-\tanh^2(Kz)]$ (as in, e.g. [6,11], with K such that $\int \mathrm{d}z\, P(z)z^2 = 1$), for which $\langle\sigma[q,z]\rangle_z = \tanh[\beta q]$ ($\beta \equiv K/T$):

$$[L_1\, p](\boldsymbol{q}) = \sum_i \frac{\partial}{\partial q_i}\left\{2p(\boldsymbol{q})\left[\frac{\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N}\sum_j \boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j \tanh[\beta q_j]\right]\right\} \tag{21}$$

$$[L_2\, p](\boldsymbol{q}) = \tilde{\eta}\sum_{ij}\frac{\partial^2}{\partial q_i \partial q_j}\left\{p(\boldsymbol{q})\left[\frac{1}{N}\sum_\mu \xi_i^\mu \xi_j^\mu \langle A^\mu[\boldsymbol{q},\boldsymbol{z}]^2\rangle_{\boldsymbol{z}}\right]\right\}. \tag{22}$$

Equation (22) describes, as expected, two types of fluctuations: $[L_2\; p](\boldsymbol{q}) = \sum_{k\ell}\partial^2_{k\ell}\left\{p(\boldsymbol{q})[M^A_{k\ell} + M^B_{k\ell}]\right\}$, one describing fluctuations induced by the randomly and sequentially drawn external information:

$$M^A_{k\ell} = \frac{\tilde{\eta}}{N}\sum_{\mu=1}^{p}\xi_k^\mu \xi_\ell^\mu \left(\Omega^\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^\mu \tanh[\beta q_j]\right)^2 \tag{23}$$

and a second one describing fluctuations induced by the decision noise[2]:

$$M^B_{k\ell} = \frac{\tilde{\eta}}{N^2}\sum_{\mu=1}^{p}\xi^\mu_k\xi^\mu_\ell\sum_i(\xi^\mu_i)^2(1-\tanh^2[\beta q_i]). \tag{24}$$

The corresponding expressions for multiplicative noise are obtained by simply replacing $\tanh[\beta q_k] \to \lambda(T)\mathrm{sgn}[q_k]$ in equations (21), (23), (24). For both noise types the second contribution (24) to the diffusion matrix vanishes in the limit $\beta \to \infty$ of deterministic decision making.

5. The generating functional

5.1. Definition and discretization

Instead of working with the Kramers–Moyal expansion, it will be more convenient and safe to return to the underlying master equation (17), which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{q}) = \frac{2p}{\tilde{\eta}}\int\frac{\mathrm{d}\hat{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q}'}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\sum_i\hat{q}_i(q_i-q'_i-\theta_i)}p_t(\boldsymbol{q}')\left\{\frac{1}{p}\sum_{\mu=1}^{p}\left\langle\mathrm{e}^{\mathrm{i}\tilde{\eta}\left(\frac{1}{\sqrt{N}}\sum_i\hat{q}_i\xi^\mu_i\right)A^\mu[\boldsymbol{q}',\boldsymbol{z}]}\right\rangle_{\boldsymbol{z}}-1\right\} \tag{25}$$

where we have introduced auxiliary driving forces $\theta_i(t)$ in order to identify response functions later. At this stage we discretize our continuous time, $t \to \ell\delta$ ($\ell = 0, 1, 2, \ldots$) with intervals $0 < \delta \ll 1$ which can be sent to zero *independent* of the limit $N \to \infty$ (since the procedure of [18], as followed here, leads to a continuous-time master equation for any N), so that the probability density of finding a *path* of microscopic states $\{\boldsymbol{q}(0), \boldsymbol{q}(\delta), \boldsymbol{q}(2\delta), \ldots\}$ can be written as

$$\mathrm{Prob}\,[\boldsymbol{q}(0), \boldsymbol{q}(\delta), \boldsymbol{q}(2\delta), \ldots] = p_0(\boldsymbol{q}(0))\prod_{t>0}\tilde{W}_t[\boldsymbol{q}(t)|\boldsymbol{q}(t-\delta)] \tag{26}$$

with

$$\tilde{W}_t[\boldsymbol{q}|\boldsymbol{q}'] = \int\frac{\mathrm{d}\hat{\boldsymbol{q}}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\sum_i\hat{q}_i(q_i-q'_i-\theta_i(t))}\left\{\frac{2\delta}{\tilde{\eta}}\sum_\mu\left\langle\mathrm{e}^{\mathrm{i}\tilde{\eta}[\frac{1}{\sqrt{N}}\sum_i\hat{q}_i\xi^\mu_i]A^\mu[\boldsymbol{q}',\boldsymbol{z}]}\right\rangle_{\boldsymbol{z}}+\left(1-\frac{2p\delta}{\tilde{\eta}}\right)\right\}. \tag{27}$$

The moment generating functional for our stochastic process is defined as

$$\begin{aligned}Z[\boldsymbol{\psi}] &= \left\langle\mathrm{e}^{\mathrm{i}\sum_t\sum_i\psi_i(t)q_i(t)}\right\rangle\\ &= \int\prod_t\left\{\mathrm{d}\boldsymbol{q}(t)\tilde{W}_t[\boldsymbol{q}(t+\delta)|\boldsymbol{q}(t)]\right\}p_0(\boldsymbol{q}(0))\,\mathrm{e}^{\mathrm{i}\sum_t\sum_i\psi_i(t)q_i(t)}.\end{aligned}$$

Derivation of the generating functional with respect to the conjugate variables $\boldsymbol{\psi}$ generates all moments of $\boldsymbol{q}$ at arbitrary times. Upon introducing the two short-hands

$$w^\mu_t = \frac{\sqrt{2}}{\sqrt{N}}\sum_i\hat{q}_i(t)\xi^\mu_i \qquad x^\mu_t = \frac{\sqrt{2}}{\sqrt{N}}\sum_i s_i(t)\xi^\mu_i$$

with $s_i(t) \equiv \sigma[q_i(t), z_i(t)]$, as well as $\mathcal{D}\boldsymbol{q} = \prod_{it}[\mathrm{d}q_i(t)/\sqrt{2\pi}]$, $\mathcal{D}\boldsymbol{w} = \prod_{\mu t}[\mathrm{d}w^\mu_t/\sqrt{2\pi}]$ and $\mathcal{D}\boldsymbol{x} = \prod_{\mu t}[\mathrm{d}x^\mu_t/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\hat{\boldsymbol{q}}$, $\mathcal{D}\hat{\boldsymbol{w}}$ and $\mathcal{D}\hat{\boldsymbol{x}}$, respectively), the

[2] Note that $\lim_{\beta\to\infty}M^B_{k\ell} = 0$, so that the $\{M^A_{k\ell}\}$ represent the fluctuations due to external information selection in the absence of decision noise. For $\beta < \infty$, however, the above causal/interpretational separation is not perfect, since the two sources of randomness will inevitably be intertwined.

generating functional takes the following form:

$$Z[\psi]=\int \mathcal{D}\boldsymbol{w}\,\mathcal{D}\hat{\boldsymbol{w}}\,\mathcal{D}\boldsymbol{x}\,\mathcal{D}\hat{\boldsymbol{x}}\;\mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu]}\prod_t\left\{1+\frac{2\delta}{\tilde{\eta}}\sum_\mu\left[\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}w_t^\mu}{\sqrt{2}}\left[\Omega^\mu+\frac{x_t^\mu}{\sqrt{2}}\right]}-1\right]\right\}$$
$$\times\int\mathcal{D}\boldsymbol{q}\,\mathcal{D}\hat{\boldsymbol{q}}\;p_0(\boldsymbol{q}(0))\left\langle\mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\right\rangle_{\boldsymbol{z}}$$
$$\times\mathrm{e}^{\mathrm{i}\sum_{ti}\left[\hat{q}_i(t)(q_i(t+\delta)-q_i(t)-\theta_i(t))+\psi_i(t)q_i(t)\right]}. \tag{28}$$

We note that, as $\delta\to 0$ (and using $\delta\sum_t=\mathcal{O}(\delta^0)$):

$$\prod_t\left\{1+\frac{2\delta}{\tilde{\eta}}\sum_\mu\left[\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}w_t^\mu}{\sqrt{2}}\left[\Omega^\mu+\frac{x_t^\mu}{\sqrt{2}}\right]}-1\right]\right\}=\mathrm{e}^{\sum_\mu\sum_t\frac{2\delta}{\tilde{\eta}}\left[\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}w_t^\mu}{\sqrt{2}}\left[\Omega^\mu+\frac{x_t^\mu}{\sqrt{2}}\right]}-1+\mathcal{O}(\delta N)\right]}.$$

Hence, if we choose $\delta\ll N^{-1}$ we obtain

$$Z[\psi]=\int \mathcal{D}\boldsymbol{w}\,\mathcal{D}\hat{\boldsymbol{w}}\,\mathcal{D}\boldsymbol{x}\,\mathcal{D}\hat{\boldsymbol{x}}\;\mathrm{e}^{\sum_{t\mu}\left[\mathrm{i}\hat{w}_t^\mu w_t^\mu+\mathrm{i}\hat{x}_t^\mu x_t^\mu+\frac{2\delta}{\tilde{\eta}}\left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega^\mu+\frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}}-1+\mathrm{o}(N^0)\right]\right]}$$
$$\times\int\mathcal{D}\boldsymbol{q}\mathcal{D}\hat{\boldsymbol{q}}\;p_0(\boldsymbol{q}(0))\left\langle\mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\right\rangle_{\boldsymbol{z}}$$
$$\times\mathrm{e}^{\mathrm{i}\sum_{ti}\left[\hat{q}_i(t)(q_i(t+\delta)-q_i(t)-\theta_i(t))+\psi_i(t)q_i(t)\right]}. \tag{29}$$

5.2. Disorder average

At this stage we carry out the disorder averages, denoted as $[\cdots]_{\mathrm{dis}}$, which involve the variables $\xi_i^\mu=\frac{1}{2}(R_{i1}^\mu-R_{i2}^\mu)$ and $\Omega^\mu=\frac{1}{2}N^{-\frac{1}{2}}\sum_j(R_{j1}^\mu+R_{j2}^\mu)$. For times which do not scale with N and for simple initial conditions of the form $p_0(\boldsymbol{q})=\prod_i p_0(q_i)$ one finds

$$[Z[\psi]]_{\mathrm{dis}}=\int \mathcal{D}\boldsymbol{w}\,\mathcal{D}\hat{\boldsymbol{w}}\,\mathcal{D}\boldsymbol{x}\,\mathcal{D}\hat{\boldsymbol{x}}\;\mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu]}$$
$$\times\int\mathcal{D}\boldsymbol{q}\,\mathcal{D}\hat{\boldsymbol{q}}\;\prod_i p_0(q_i(0))\;\mathrm{e}^{\mathrm{i}\sum_{ti}\left[\hat{q}_i(t)(q_i(t+\delta)-q_i(t)-\theta_i(t))+\psi_i(t)q_i(t)\right]}$$
$$\times\left[\left\langle\prod_\mu\mathrm{e}^{\frac{2\delta}{\tilde{\eta}}\sum_t\left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega^\mu+\frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}}-1+\mathrm{o}(N^0)\right]-\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_i\xi_i\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\right\rangle_{\boldsymbol{z}}\right]_{\mathrm{dis}}. \tag{30}$$

We concentrate on the term in (30) with the disorder averages. These are similar to but generally more difficult than those calculated in [20] (to which they reduce only for $\tilde{\eta}\to 0$). As in [20] they automatically generate the dynamic order parameters $C_{tt'}=N^{-1}\sum_i s_i(t)s_i(t')$, $K_{tt'}=N^{-1}\sum_i s_i(t)\hat{q}_i(t')$, and $L_{tt'}=N^{-1}\sum_i\hat{q}_i(t)\hat{q}_i(t')$ and their conjugates

$$\prod_\mu[\ldots]_{\mathrm{dis}}=\prod_\mu\left\{\int\frac{\mathrm{d}\Omega\,\mathrm{d}\hat{\Omega}}{2\pi}\;\mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega+\frac{2\delta}{\tilde{\eta}}\sum_t\left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega+\frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}}-1+\mathrm{o}(N^0)\right]}\right.$$
$$\left.\times\prod_i\left[\mathrm{e}^{-\frac{\mathrm{i}R_{i1}^\mu}{\sqrt{2N}}\left[\hat{\Omega}/\sqrt{2}+\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right]-\frac{\mathrm{i}R_{i2}^\mu}{\sqrt{2N}}\left[\hat{\Omega}/\sqrt{2}-\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right]}\right]_{\mathrm{dis}}\right\}$$
$$=\prod_\mu\left\{\int\frac{\mathrm{d}\Omega\,\mathrm{d}\hat{\Omega}}{2\pi}\;\mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega+\frac{2\delta}{\tilde{\eta}}\sum_t\left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega+\frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}}-1\right]+\mathrm{o}(N^0)}\right.$$
$$\left.\times\;\mathrm{e}^{-\frac{1}{4N}\sum_i\left[\left(\frac{\hat{\Omega}}{\sqrt{2}}+\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right)^2+\left(\frac{\hat{\Omega}}{\sqrt{2}}-\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right)^2\right]}\right\}$$

$$
\begin{aligned}
&= \prod_\mu \left\{ \int \frac{\mathrm{d}\Omega\,\mathrm{d}\hat{\Omega}}{2\pi}\; \mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega - \frac{1}{4}\hat{\Omega}^2 + \frac{2\delta}{\tilde{\eta}}\sum_t \left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega + \frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}} - 1\right] + \mathrm{o}(N^0)} \right.\\
&\quad \left. \times\, \mathrm{e}^{-\frac{1}{2N}\sum_i \left[\sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]\right]^2} \right\}\\
&= \prod_\mu \left\{ \int \frac{\mathrm{d}\Omega}{\sqrt{\pi}}\; \mathrm{e}^{-\Omega^2 + \frac{2\delta}{\tilde{\eta}}\sum_t \left[\mathrm{e}^{\mathrm{i}\tilde{\eta}w_t^\mu\left[\Omega + \frac{x_t^\mu}{\sqrt{2}}\right]/\sqrt{2}} - 1\right] + \mathrm{o}(N^0)} \right.\\
&\quad \left. \times\, \mathrm{e}^{-\frac{1}{2}\sum_{tt'}\left[\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu + 2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu + \hat{x}_t^\mu C_{tt'}\hat{x}_{t'}^\mu\right]} \right\}. \qquad (31)
\end{aligned}
$$

Insertion into (30), followed by the isolation of the order parameters via δ-distributions (whose integral representations generate the conjugate order parameters) then gives

$$
[Z[\psi]]_{\mathrm{dis}} = \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\; \mathrm{e}^{N[\Psi + \Phi + \Omega + \mathrm{o}(N^0)]}. \qquad (32)
$$

The $\mathrm{o}(N^0)$ term in the exponent is independent of the fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$. Upon choosing an appropriate scaling with δ of the conjugate integration variables (namely, $(\hat{x}, \hat{w}) \to \delta(\hat{x}, \hat{w})$ and $(\hat{C}, \hat{K}, \hat{L}) \to \delta^2(\hat{C}, \hat{K}, \hat{L})$, to guarantee the existence of a proper $\delta \to 0$ limit of the various time summations), the three relevant exponents in (32) are given by the following expressions:

$$
\Psi = \mathrm{i}\delta^2 \sum_{tt'} [\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}] \qquad (33)
$$

$$
\begin{aligned}
\Phi = \alpha \log\Bigg[&\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\; \mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}[\hat{w}_t L_{tt'}\hat{w}_{t'} + 2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}]}\\
&\times \mathrm{e}^{\mathrm{i}\delta\sum_t[\hat{w}_t w_t + \hat{x}_t x_t]} \int Du\; \mathrm{e}^{\frac{2\delta}{\tilde{\eta}}\sum_t\left[\mathrm{e}^{\frac{1}{2}\mathrm{i}\tilde{\eta}w_t[u + x_t]} - 1\right]} \Bigg] \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
\Omega = \frac{1}{N}\sum_i \log\Bigg\langle &\int \mathcal{D}q\mathcal{D}\hat{q}\; p_0(q(0))\; \mathrm{e}^{\mathrm{i}\delta\sum_t\left[\hat{q}(t)\left(\frac{q(t+\delta) - q(t)}{\delta} - \delta^{-1}\theta_i(t)\right) + \delta^{-1}\psi_i(t)q(t)\right]}\\
&\times \mathrm{e}^{-\mathrm{i}\delta^2\sum_{tt'}[\hat{C}_{tt'}s(t)s(t') + \hat{K}_{tt'}s(t)\hat{q}(t') + \hat{L}_{tt'}\hat{q}(t)\hat{q}(t')]} \Bigg\rangle_z \qquad (35)
\end{aligned}
$$

with the standard abbreviation of the Gaussian measure $Du = (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}u^2}\,\mathrm{d}u$. The average $\langle \ldots \rangle_z$ has now been reduced to a single site one: $\langle g[z_1, z_2, \ldots] \rangle_z = \int \prod_t [\mathrm{d}z_t\, P(z_t)]\; g[z_1, z_2, \ldots]$. Following [20] we have also introduced the short-hands $\mathcal{D}q = \prod_t [\mathrm{d}q\,(t)/\sqrt{2\pi}]$, $\mathcal{D}w = \prod_t [\mathrm{d}w_t/\sqrt{2\pi}]$, $\mathcal{D}x = \prod_t [\mathrm{d}x_t/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\hat{q}$, $\mathcal{D}\hat{w}$ and $\mathcal{D}\hat{x}$).

6. The saddle-point equations

6.1. Derivation of saddle-point equations

We can now evaluate (32) by saddle-point integration, in the limit $N \to \infty$, and provided the order parameter functions depend asymptotically on their two time arguments in a sufficiently smooth (i.e. N-independent) way. We define $G_{tt'} = -\mathrm{i}K_{tt'}$. Taking derivatives with respect to the generating fields and using the normalization $[Z[\mathbf{0}]]_{\mathrm{dis}} = 1$ then gives (at the physical saddle-point) the usual relations

$$
C_{tt'} = \lim_{N\to\infty} \frac{1}{N}\sum_i \left[\langle s_i(t)s_i(t')\rangle\right]_{\mathrm{dis}} \qquad (36)
$$

$$G_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial}{\partial\theta_i(t')} [\langle s_i(t)\rangle]_{\rm dis} \tag{37}$$

$$L_{tt'} = 0. \tag{38}$$

Putting $\psi_i(t) = 0$ (they are no longer needed) and $\theta_i(t) = \delta \cdot \tilde{\theta}(t)$ then simplifies (35) to

$$\Omega = \log \int \mathcal{D}q\, \mathcal{D}\hat{q}\;\; p_0(q(0))\, \mathrm{e}^{-\mathrm{i}\delta^2 \sum_{tt'} \hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}$$
$$\times \left\langle \mathrm{e}^{\mathrm{i}\delta \sum_t \hat{q}(t)\left[\frac{q(t+\delta)-q(t)}{\delta} - \tilde{\theta}(t) - \delta\sum_{t'} \hat{K}_{t't} s(t')\right] - \mathrm{i}\delta^2 \sum_{tt'} s(t)\hat{C}_{tt'} s(t')} \right\rangle_z \tag{39}$$

in which now $s(t) = \sigma[q(t), z_t]$. Extremization of the extensive exponent $\Psi + \Phi + \Omega$ of (32) with respect to $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ gives the remaining saddle-point equations

$$C_{tt'} = \langle s(t)s(t')\rangle_\star \qquad G_{tt'} = \frac{\partial\langle s(t)\rangle_\star}{\delta\partial\tilde{\theta}(t')} \tag{40}$$

$$\hat{C}_{tt'} = \frac{\mathrm{i}}{\delta^2}\frac{\partial\Phi}{\partial C_{tt'}} \qquad \hat{K}_{tt'} = \frac{\mathrm{i}}{\delta^2}\frac{\partial\Phi}{\partial K_{tt'}} \qquad \hat{L}_{tt'} = \frac{\mathrm{i}}{\delta^2}\frac{\partial\Phi}{\partial L_{tt'}}. \tag{41}$$

The effective single-trader averages $\langle\ldots\rangle_\star$, generated by taking derivatives of (35), are defined as

$$\langle f[\{q, s\}]\rangle_\star = \frac{\int \mathcal{D}q\; \langle M[\{q, s\}] f[\{q, s\}]\rangle_z}{\int \mathcal{D}q\; \langle M[\{q, s\}]\rangle_z} \tag{42}$$

$$M[\{q, s\}] = p_0(q(0))\, \mathrm{e}^{-\mathrm{i}\delta^2 \sum_{tt'} s(t)\hat{C}_{tt'} s(t')} \int \mathcal{D}\hat{q}\; \mathrm{e}^{-\mathrm{i}\delta^2 \sum_{tt'} \hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}$$
$$\times\, \mathrm{e}^{\mathrm{i}\delta \sum_t \hat{q}(t)\left[\frac{q(t+\delta)-q(t)}{\delta} - \tilde{\theta}(t) - \delta\sum_{t'} \hat{K}_{t't} s(t')\right]}. \tag{43}$$

Upon elimination of the trio $\{\hat{C}, \hat{K}, \hat{L}\}$ via (41) we obtain exact closed equations for the disorder-averaged correlation and response functions in the $N \to \infty$ limit: equations (40), with the effective single-trader measure (43).

6.2. Simplification of saddle-point equations

The introduction of on-line evolution and decision noise into the dynamics has affected the terms Φ (34) and Ω (39), compared to the analysis in [20]. In order to proceed we now have to work out the term Φ (34) further:

$$\Phi = \alpha \log \int Du\, \mathcal{D}w\, \mathcal{D}x\; \phi[\{x, w\}, u]$$
$$\times \int \mathcal{D}\hat{w}\, \mathcal{D}\hat{x}\; \mathrm{e}^{-\frac{1}{2}\delta^2 \sum_{tt'}[\hat{w}_t L_{tt'}\hat{w}_{t'} + 2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}] + \mathrm{i}\delta\sum_t [\hat{w}_t w_t + \hat{x}_t x_t]} \tag{44}$$

where

$$\phi[\{x, w\}, u] = \exp\left[\frac{2\delta}{\tilde{\eta}} \sum_t \left[\mathrm{e}^{\frac{1}{2}\mathrm{i}\tilde{\eta} w_t [u + x_t]} - 1\right]\right]. \tag{45}$$

Upon expanding the 'inner' exponential in this expression in a power series, we can write ϕ as an average of the form

$$\phi[\{x, w\}, u] = \left\langle \exp\left[\tfrac{1}{2}\mathrm{i}\tilde{\eta} \sum_t n_t w_t [u + x_t]\right]\right\rangle_n \tag{46}$$

where $\langle\cdot\rangle_n$ is defined as

$$\langle f[n_1, n_2, \ldots]\rangle_n \equiv \sum_{n_1, n_2 \ldots \geqslant 0} \left[\prod_s P_{2\delta/\tilde{\eta}}[n_s]\; f[n_1, n_2, \ldots]\right] \tag{47}$$

with the Poisson distribution $P_a[\ell] = \frac{1}{\ell!}e^{-a}a^\ell$. The first two moments of the distribution are given by $\langle n_t\rangle_n = 2\delta/\tilde{\eta}$ and $\langle n_t^2\rangle_n = (2\delta/\tilde{\eta})^2 + 2\delta/\tilde{\eta}$. We now insert (46) into (44), followed by integration over u, $\{x\}$, $\{\hat{x}\}$ and $\{w\}$ (in precisely that order). This gives, with $K = \mathrm{i}G$, with the diagonal matrix $E_{tt'} = \frac{1}{2}\tilde{\eta}n_t\delta_{tt'}$, and with the matrix D whose entries are defined as $D_{tt'} = 1 + C_{tt'}$:

$$\Phi = \alpha\log\left\langle \mathrm{Det}^{-\frac{1}{2}}[EDE]\int \mathcal{D}\hat{w}\; \mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}\hat{w}_t[L+(\mathbb{I}+G^\dagger E)(EDE)^{-1}(\mathbb{I}+EG)]_{tt'}\hat{w}_{t'}}\right\rangle_n \tag{48}$$

(modulo an irrelevant constant). Taking the derivative of Φ with respect to the matrix elements $\{L_{tt'}, C_{tt'}, G_{tt'}\}$, followed by setting $L\to 0$, gives (using the causality property $G_{tt'} = 0$ for $t \leqslant t'$, which guarantees that $\mathrm{Det}(\mathbb{I}+EG) = 1$):

$$\lim_{L\to 0}\frac{\partial\Phi}{\partial L_{tt'}} = -\frac{1}{2}\alpha\langle[(\mathbb{I}+EG)^{-1}EDE(\mathbb{I}+G^\dagger E)^{-1}]_{tt'}\rangle_n \tag{49}$$

$$\lim_{L\to 0}\frac{\partial\Phi}{\partial C_{tt'}} = 0 \tag{50}$$

$$\lim_{L\to 0}\frac{\partial\Phi}{\partial G_{tt'}} = -\alpha\langle[(\mathbb{I}+EG)^{-1}E]_{t't}\rangle_n. \tag{51}$$

According to (41) this gives for the conjugate order parameters

$$\hat{L}_{tt'} = -\tfrac{1}{2}\mathrm{i}\alpha\Sigma_{tt'} \tag{52}$$

$$\hat{C}_{tt'} = 0 \tag{53}$$

$$\hat{K}_{tt'} = -\alpha R_{t't} \tag{54}$$

with

$$\Sigma = \frac{1}{\delta^2}\langle(\mathbb{I}+EG)^{-1}EDE(\mathbb{I}+G^\dagger E)^{-1}\rangle_n \tag{55}$$

$$R = \frac{1}{\delta^2}\langle(\mathbb{I}+EG)^{-1}E\rangle_n. \tag{56}$$

6.3. Evaluation of poisson averages

Finally, it turns out that in these latter two matrices (55), (56) the averages over the $\{n_t\}$ can be performed exactly, by using causality. We first deal with (56) (which is simpler):

$$R_{tt'} = \frac{1}{\delta^2}\sum_{\ell\geqslant 0}(-1)^\ell\langle[(EG)^\ell E]_{tt'}\rangle_n.$$

We note that each of the terms in the expansion gives factorized averages due to $G_{ss'} = 0$ for $s \leqslant s'$ (using only that $\langle n_s\rangle_n = 2\delta/\tilde{\eta}$ for any s):

$$\begin{aligned}\langle[(EG)^\ell E]_{tt'}\rangle_n &= (\tfrac{1}{2}\tilde{\eta})^{\ell+1}\sum_{s_1,\ldots,s_{\ell-1}}\langle n_t G_{ts_1}n_{s_1}G_{s_1s_2}n_{s_2}G_{s_2s_3}\ldots n_{s_{\ell-1}}G_{s_{\ell-1}t'}n_{t'}\rangle_n\\ &= (\tfrac{1}{2}\tilde{\eta})^{\ell+1}\sum_{t>s_1>\cdots>s_{\ell-1}>t'}G_{ts_1}G_{s_1s_2}\ldots G_{s_{\ell-1}t'}\langle n_t n_{s_1}\ldots n_{s_{\ell-1}}n_{t'}\rangle_n\\ &= \delta^{\ell+1}G^\ell_{tt'}\end{aligned}$$

giving the simple result

$$R = \frac{1}{\delta}[\mathbb{I}+\delta G]^{-1}. \tag{57}$$

We note that in calculating (57) knowledge of only the first moment of the $\{n_t\}$ distribution was required.

Next we turn to the noise covariance matrix

$$\Sigma_{tt'} = \frac{1}{\delta^2} \sum_{\ell\ell' \geqslant 0} (-1)^{\ell+\ell'} \langle [(EG)^\ell EDE(G^\dagger E)^{\ell'}]_{tt'} \rangle_n. \tag{58}$$

Now, again due to causality, only averages of terms with at most *two* n-variables with the same time index can occur. We have to take into account the extra contributions coming from all possible time pairings. Using $\langle n_s^2 \rangle = \langle n_s \rangle^2 [1 + \tilde{\eta}/2\delta]$ we derive

$$\begin{aligned}
\langle [(EG)^\ell EDE(G^\dagger E)^{\ell'}]_{s_0 s_0'} \rangle_n &= \sum_{s_\ell s_\ell'} D_{s_\ell s_\ell'} \langle [(EG)^\ell E]_{s_0 s_\ell} [(EG)^{\ell'} E]_{s_0' s_\ell'} \rangle_n \\
&= (\tfrac{1}{2}\tilde{\eta})^{\ell+\ell'+2} \sum_{s_0 > \cdots > s_\ell} \sum_{s_0' > \cdots > s_\ell'} D_{s_\ell s_\ell'} \, G_{s_0 s_1} \ldots G_{s_{\ell-1} s_\ell} \, G_{s_0' s_1'} \ldots G_{s_{\ell'-1}' s_\ell'} \\
&\quad \times \langle n_{s_0} \ldots n_{s_\ell} n_{s_0'} \ldots n_{s_\ell'} \rangle_n \\
&= \delta^{\ell+\ell'+2} \sum_{s_0 > \cdots > s_\ell} \sum_{s_0' > \cdots > s_\ell'} D_{s_\ell s_\ell'} \left[1 + \frac{\tilde{\eta}}{2\delta} \right]^{\sum_{i=0}^{\ell} \sum_{j=0}^{\ell'} \delta_{s_i s_j'}} \\
&\quad \times G_{s_0 s_1} \ldots G_{s_{\ell-1} s_\ell} \, G_{s_0' s_1'} \ldots G_{s_{\ell'-1}' s_\ell'} \\
&= \delta^{\ell+\ell'+2} \sum_{s_0 > \cdots > s_\ell} \sum_{s_0' > \cdots > s_\ell'} D_{s_\ell s_\ell'} \prod_{i=0}^{\ell} \prod_{j=0}^{\ell'} \left[1 + \delta_{s_i s_j'} \frac{\tilde{\eta}}{2\delta} \right] \\
&\quad \times G_{s_0 s_1} \ldots G_{s_{\ell-1} s_\ell} \, G_{s_0' s_1'} \ldots G_{s_{\ell'-1}' s_\ell'}
\end{aligned} \tag{59}$$

where the factor $\sum_{i=0}^{\ell} \sum_{j=0}^{\ell'} \delta_{s_i s_j'}$ counts the number of pairings occurring in the two types of time arguments (i.e. those with primes, and those without). We note that to find expression (59) knowledge of only the first two moments of the $\{n_t\}$-distribution was required.

7. The effective single-agent process

7.1. Discretized single-agent process

Upon inserting the results (52)–(54) we now find our effective single-trader measure $M[\{q, s\}]$ of (43) reducing further to the following expression (modulo a constant pre-factor reflecting normalization, which is independent of the decision noise variables $\{z(t)\}$):

$$\begin{aligned}
M[\{q, s\}] = p_0(q(0)) \int \mathcal{D}\eta \, \mathrm{e}^{-\frac{1}{2} \sum_{tt'} \eta(t) \Sigma_{tt'}^{-1} \eta(t')} \\
\times \prod_t \delta \left[\frac{q(t+\delta) - q(t)}{\delta} - \tilde{\theta}(t) + \alpha\delta \sum_{t'} R_{tt'} s(t') - \sqrt{\alpha}\eta(t) \right]
\end{aligned} \tag{60}$$

with $R_{tt'} = \delta^{-1} [\mathbb{1} + \delta G]_{tt'}^{-1}$ (57). This describes a single-agent process of the form

$$\frac{q(t+\delta) - q(t)}{\delta} = \tilde{\theta}(t) - \alpha\delta \sum_{t'} R_{tt'} \sigma[q(t'), z(t')] + \sqrt{\alpha}\eta(t). \tag{61}$$

Causality ensures that $R_{tt'} = 0$ for $t' > t$. The variable z_t represents the original single-trader decision noise, with $\langle z(t) \rangle_z = 0$ and $\langle z(t) z(t') \rangle_z = \delta_{tt'}$, and $\eta(t)$ is a disorder-generated Gaussian noise with zero mean and with temporal correlations given by (58), (59).

7.2. Continuous time limit

We can now take the limit $\delta \to 0$ and restore continuous time. The bookkeeping of δ-terms is found to come out right, with partial derivatives with respect to the perturbation fields converting into functional derivatives, with time summations converting into integrals, and with matrices converting into integral operators (with the usual convention $\frac{1}{\delta}\mathbb{I}_{tt'} \to \delta[t-t']$). Upon making the ansatz that our order parameters are smooth functions of time, we then lose the remaining microscopic variables $\{z(t)\}$ in the retarded self-interaction term (which are automatically converted into averages over their distribution) and end up with the effective single-trader problem

$$\frac{\mathrm{d}}{\mathrm{d}t}q(t) = \tilde{\theta}(t) - \alpha\int_0^t \mathrm{d}t'\, R(t,t')\langle\sigma[q(t'),z]\rangle_z + \sqrt{\alpha}\eta(t). \tag{62}$$

The correlation- and response functions (36), (37) are the dynamic order parameters of the problem, and must now be solved self-consistently from the following closed equations:

$$C(t,t') = \big\langle\langle\sigma[q(t),z]\rangle_z\, \langle\sigma[q(t'),z]\rangle_z\big\rangle_\star \tag{63}$$

$$G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\,\langle\langle\sigma[q(t),z]\rangle_z\rangle_\star \tag{64}$$

(for $t \neq t'$). The brackets $\langle\ldots\rangle_\star$ now refer to the stochastic process (62) (which no longer involves the $\{z(t)\}$, only their distribution). Note that the correlation and response functions are generally discontinuous at $t = t'$, namely $C(t,t) = 1$ and $G(t,t) = 0$ (in our derivation we have used the Itô convention). For additive noise with $P(z) = \frac{1}{2}K[1-\tanh^2(Kz)]$ (as in e.g. [6, 11], with K such that $\int \mathrm{d}z\, P(z)z^2 = 1$), for instance, one has $\langle\sigma[q,z]\rangle_z = \tanh[\beta q]$ with inverse 'temperature' $\beta \equiv K/T$. While for multiplicative noise one finds $\langle\sigma[q,z]\rangle_z = \lambda(T)\mathrm{sgn}\,(q)$ with $\lambda(T) = \int\mathrm{d}z\, P(z)\mathrm{sgn}\,[1+Tz]$.

What remains is to take the continuous time limit in our expressions for the effective noise covariance and the retarded self-interaction kernel. The latter, defined by (57), simply becomes

$$\begin{aligned} R(t,t') &= \lim_{\delta\to 0}\frac{1}{\delta}[\mathbb{I}+\delta G]^{-1}_{tt'} \\ &= \delta(t-t') + \sum_{\ell>0}(-1)^\ell G^\ell(t,t') \end{aligned} \tag{65}$$

with the usual definition for multiplication of the continuous time kernels $G^{\ell+1}(t,t') = \int\mathrm{d}t''\, G^\ell(t,t'')G(t'',t')$. The continuous time limit of the noise covariance kernel (58) becomes

$$\begin{aligned} \Sigma(s_0,s_0') &= \sum_{\ell\ell'\geqslant 0}(-1)^{\ell+\ell'}\lim_{\delta\to 0}\delta^{\ell+\ell'}\sum_{s_0>\cdots>s_\ell\geqslant 0}\;\sum_{s_0'>\cdots>s_{\ell'}'\geqslant 0}\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\left[1+\delta_{s_i s_j'}\frac{\tilde{\eta}}{2\delta}\right] \\ &\quad\times[1+C(s_\ell,s_{\ell'}')]G(s_0,s_1)\ldots G(s_{\ell-1},s_\ell)G(s_0',s_1')\ldots G(s_{\ell'-1}',s_{\ell'}') \\ &= \sum_{\ell\ell'\geqslant 0}(-1)^{\ell+\ell'}\int_0^\infty \mathrm{d}s_1\ldots\mathrm{d}s_\ell\,\mathrm{d}s_1'\ldots\mathrm{d}s_{\ell'}'\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\left[1+\tfrac{1}{2}\tilde{\eta}\delta(s_i-s_j')\right] \\ &\quad\times[1+C(s_\ell,s_{\ell'}')]G(s_0,s_1)\ldots G(s_{\ell-1},s_\ell)G(s_0',s_1')\ldots G(s_{\ell'-1}',s_{\ell'}'). \end{aligned} \tag{66}$$

8. Stationary state in the ergodic regime

In this section we study the long-time limit of the effective single-agent process (62) in the regime where the following three conditions are met: time translation invariance, i.e.

$$\lim_{t\to\infty} C(t+\tau,t) = C(\tau) \qquad \lim_{t\to\infty} G(t+\tau,t) = G(\tau) \tag{67}$$

a finite integrated response (or static susceptibility), i.e.

$$\lim_{t\to\infty}\int \mathrm{d}t'\, G(t,t') = \chi < \infty \tag{68}$$

and weak long-term memory [22], i.e.

$$\lim_{t\to\infty}\int_0^{t_w} \mathrm{d}t'\, G(t,t') = 0 \text{ for any fixed } t_w. \tag{69}$$

Together these three conditions ensure that also the retarded self-interaction R and the noise covariance matrix Σ will become time translation invariant: $\lim_{t\to\infty} R(t+\tau,t) = R(\tau)$ and $\lim_{t\to\infty} \Sigma(t+\tau,t) = \Sigma(\tau)$. We introduce the following notation for long-time averages: $\overline{f} = \lim_{\tau\to\infty}\tau^{-1}\int_0^\tau \mathrm{d}t\, f(t)$. Upon assuming the above three conditions to hold, we can calculate the long-time average for the single-agent process (62), giving

$$\overline{\mathrm{d}q/\mathrm{d}t} = \overline{\theta} - \frac{\alpha}{1+\chi}\overline{\langle\sigma\rangle_z} + \sqrt{\alpha}\,\overline{\eta}. \tag{70}$$

It has been noted and exploited earlier (in e.g. [10] and [20]) that agents can be divided into two categories: frozen agents who get completely fixed on one specific strategy, and fickle agents who always continue to alternate their strategies. The first type of agent will have a non-zero average preference velocity, $\overline{\mathrm{d}q/\mathrm{d}t} \neq 0$, while for the agents in the latter group one has $\overline{\mathrm{d}q/\mathrm{d}t} = 0$. For fickle agents it follows from equation (70) that $\overline{\langle\sigma\rangle_z} = (1+\chi)(\overline{\theta}+\sqrt{\alpha}\overline{\eta})/\alpha$. This implies that a necessary condition for an agent to be fickle is

$$\text{fickle agents:}\quad |\overline{\theta}+\sqrt{\alpha}\overline{\eta}| \leqslant \frac{\alpha}{1+\chi}|\langle\sigma[\pm\infty,z]\rangle_z| \equiv \gamma. \tag{71}$$

Note that $|\langle\sigma[\pm\infty,z]\rangle_z| = \lambda(T)$ for multiplicative noise, and that $|\langle\sigma[\pm\infty,z]\rangle_z| = 1$ for additive noise. With the conventions $\lambda = \lambda(T)$ for multiplicative noise and $\lambda = 1$ for additive noise, we cover both cases by writing $\gamma = \lambda\alpha/(1+\chi)$. If an agent is frozen, the preference velocity $\overline{\mathrm{d}q/\mathrm{d}t}$ must have the same sign as $q(t)$, and thus also the same sign as $\overline{\langle\sigma\rangle_z}$. This implies

$$\text{frozen agents :}\quad |\overline{\theta}+\sqrt{\alpha}\overline{\eta}| \geqslant \frac{\alpha}{1+\chi}|\overline{\langle\sigma\rangle_z}| = \gamma. \tag{72}$$

Since the two conditions (71), (72) are complementary, they are are not only necessary but also sufficient for characterizing agents as either fickle (71) or frozen (72). The asymptotic behaviour of the agent is thus completely determined by the persistent noise $\overline{\eta}$, which is a Gaussian random variable with zero mean and a variance given by

$$\langle\overline{\eta}^2\rangle_\star = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t'\,\Sigma(t') = \frac{1+c}{(1+\chi)^2} \tag{73}$$

in which we find the persistent auto-correlation $c = \lim_{\tau\to\infty}\tau^{-1}\int_0^\tau \mathrm{d}t\, C(t)$. The latter observable can be expressed in terms of the integrated response χ, the market's control parameter α, and the fraction of frozen agents ϕ. To do this we first separate the expression for c into frozen and fickle contributions, by using the conditions (71), (72) (we may set $\overline{\theta} = 0$):

$$\begin{aligned} c &= \left\langle \overline{\langle\sigma\rangle_z}^2\,\Theta(|\sqrt{\alpha}\overline{\eta}|-\gamma)\right\rangle_\star + \left\langle \overline{\langle\sigma\rangle_z}^2\,\Theta(\gamma-|\sqrt{\alpha}\overline{\eta}|)\right\rangle_\star \\ &= \lambda^2\left\langle\Theta(|\sqrt{\alpha}\overline{\eta}|-\gamma)\right\rangle_\star + \left(\frac{1+\chi}{\sqrt{\alpha}}\right)^2\left\langle\overline{\eta}^2\Theta(\gamma-|\sqrt{\alpha}\overline{\eta}|)\right\rangle_\star \end{aligned} \tag{74}$$

where Θ denotes the step-function. Only Gaussian integrals remain (defining the distribution of $\overline{\eta}$). To compactify the final result it is convenient to introduce

$$y = \frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}. \tag{75}$$

The fraction of frozen agents ϕ and the persistent correlation c can now be written as

$$\phi = 1 - \mathrm{erf}\,(y) \tag{76}$$

$$c = \lambda^2 \left(\phi(y) + \frac{2}{y^2} \int_0^y \frac{\mathrm{d}x}{\sqrt{\pi}}\, x^2 \mathrm{e}^{-x^2} \right). \tag{77}$$

Under the conditions (67)–(69) we can calculate the static susceptibility without knowing the non-persistent parts of the response- and correlation functions. The reason for this is that, asymptotically, the frozen agents will not change their preferences when subjected to an infinitesimal external perturbation field, whereas the fickle agents will react linearly to such a field. In formulae

$$\chi = \lim_{t\to\infty} \int_0^t \mathrm{d}t' \, \frac{\delta}{\delta\tilde{\theta}(t')} \langle\langle \sigma[q(t), z] \rangle_z \rangle_\star = \frac{\partial}{\partial\overline{\theta}} \left\langle \overline{\langle\sigma\rangle_z} \right\rangle_\star = \frac{1}{\alpha}(1-\phi)(1+\chi).$$

Or, equivalently

$$\chi = \frac{1-\phi}{\alpha - (1-\phi)}. \tag{78}$$

The equations (75), (77) and (78) completely specify the stationary behaviour of the system. They are exact for the on-line MG. We will not give a detailed exposition of the behaviour of the solutions of the equations here, because they have been presented and discussed at length in earlier papers. The interested reader is referred to [12] and [20] in particular.

However, it is appropriate at this point to discuss the status of our assumptions (67)–(69). Equation (78) tells us that χ is positive and finite for $\alpha > 1 - \phi$, and will diverge at $\alpha = 1 - \phi$. Numerical solution of (75), (77), (78) shows that this happens at $\alpha = \alpha_c(T)$, where $\alpha_c(0) \approx 0.3374$, and $\alpha_c(T)$ tends to 0 as $T \to \infty$ (see [21] for the phase diagram in the (α, T) plane). For $\alpha > \alpha_c(T)$ simulations and theory are found to be in perfect agreement, and there is no evidence that the assumptions (67)–(69) do not hold. Below $\alpha_c(T)$, however, the system's behaviour is known to depend strongly on initial conditions [15, 17, 20]. This indicates that most likely not only condition (68) (finite integrated response) ceases to hold, but also (69) (weak long-term memory). These two conditions may seem very similar; however, work in progress on a version of the MG where agents try to correct for their own impact on the market [23] indicates that it is possible to have a long-term memory *and* a finite integrated response [24]. For the original MG as presented here we have found no evidence that this can happen. Hence, equations (77) and (78) can for $\alpha > \alpha_c(T)$ be regarded as an exact and complete description of the persistent order parameters of the MG.

9. The volatility

9.1. Definitions and exact relations

An important measure of the efficiency of a market is the average mismatch of buyers and sellers. In the case of the MG this is given by the magnitude of the total bid $A(\ell)$. Since in the present model the long-term average $\overline{A} = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell\leqslant L} A(\ell)$ vanishes, the appropriate measure here is the volatility σ, which measures the size of the fluctuations of A (see equation (8)) in the stationary state

$$\sigma^2 = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} A(\ell)^2 = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} \{A^{\mu(\ell)}[\boldsymbol{q}(\ell), \boldsymbol{z}(\ell)]\}^2. \tag{79}$$

If the limit $L \to \infty$ is taken before the thermodynamic limit $N \to \infty$, then the volatility will be self-averaging with respect to the realization of the presentation of the patterns $\{\mu(\ell)\}$, with respect to the realization of the decision noise $\{z(\ell)\}$, and with respect to the realization of the quenched disorder variables $\boldsymbol{\Omega}$ and $\boldsymbol{\xi}$. Hence, in the thermodynamic limit, the average volatility (over the above sources of randomness) will be identical to the single sample volatility. In the continuous-time version of the process, i.e. after the introduction of Poisson-distributed iteration durations, one may write

$$\sigma^2 = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t \left[\frac{1}{p}\sum_\mu \langle\langle A^\mu[\boldsymbol{q}(t), \boldsymbol{z}(t)]^2\rangle_z\rangle\right]_{\mathrm{dis}}. \tag{80}$$

Here the average without subscripts refers to the full stochastic process (including both the randomness in the selection of the external information as well as that induced by the decision noise). The approach followed in this paper in fact allows us to study a more general object than the volatility

$$\Xi(t, t') = \frac{1}{p}\sum_\mu \left[\langle\langle A^\mu[\boldsymbol{q}(t), \boldsymbol{z}(t)]A^\mu[\boldsymbol{q}(t'), \boldsymbol{z}(t')]\rangle_z\rangle\right]_{\mathrm{dis}}. \tag{81}$$

From (81) the volatility follows as

$$\sigma^2 = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t\ \Xi(t, t). \tag{82}$$

In contrast to (80), the quantity (81) also describes transient bid fluctuations and their correlations. The quantity (81) is calculated via a simple adaptation of the corresponding calculation in [20] (to which we refer for full details). One introduces the observables $A_t^\mu = \Omega^\mu + x_t^\mu/\sqrt{2}$ explicitly into the calculation of the generating functional using a delta function (written in integral form, which generates the conjugate variables $\hat{A}_t^\mu$), similar to the introduction of x_t^μ and w_t^μ. After having done the integral over Ω^μ one then finds $A_t^\mu = (u + x_t)/\sqrt{2}$. If we apply this modification to equations (44), (46), and perform the Gaussian integrals over u, x, $\hat{x}$, w, $\hat{w}$ and $\hat{A}$ (in this order) we are left with (modulo an irrelevant constant):

$$\Phi = \frac{1}{N}\sum_\mu \log\left\langle \int \mathcal{D}A^\mu \exp\left[-\sum_{tt'} A_t^\mu\left[(\mathbb{I} + GE)^\dagger D^{-1}(\mathbb{I} + GE)\right]_{tt'} A_{t'}^\mu\right]\right\rangle_n.$$

From this it follows that the covariance matrix (80) is half the inverse of the matrix appearing in the above exponential between A_t^μ and $A_{t'}^\mu$:

$$\Xi_{tt'} = \tfrac{1}{2}\langle[(\mathbb{I} + GE)^{-1}D(\mathbb{I} + EG^\dagger)^{-1}]_{tt'}\rangle_n. \tag{83}$$

In this expression one can carry out explicitly the average over the variables $\{n_t\}$, similar to the calculation of the noise covariance matrix (66) (to which (81) is found to be similar but, in contrast to the batch cases studied in [20, 21], not proportional). Upon subsequently taking the continuous time limit $\delta \to 0$ this leads to an exact expression for the generalized volatility matrix (81), at any combination of times (t, t'), in terms of our dynamical order parameters $C(t, t')$ and $G(t, t')$.

9.2. *Expression for the volatility in terms of persistent order parameters*

Since in practice it is very difficult to solve the order parameter equations (63), (64) for finite temporal separations, it would be helpful to find an expression for (83) (and hence also the volatility) which involves persistent order parameters only. In the remainder of this section we discuss procedures to achieve this in the ergodic regime $\alpha > \alpha_c(T)$. First we turn to the long-time correlation in bids. If t and t' are sufficiently separated and conditions (67)–(69) hold (so that G decays effectively on finite time-scales), the Poisson average in (83) factorizes over the two E matrices, so that in the continuous time limit the long-term bid correlations reduce to

$$\begin{aligned}\Xi(\infty) &= \lim_{\tau\to\infty}\lim_{t\to\infty}\Xi(t+\tau,t)\\ &= \lim_{\delta\to 0}\lim_{\tau\to\infty}\lim_{t\to\infty}\tfrac{1}{2}\big[(\mathbb{I}+\delta G)^{-1}D(\mathbb{I}+\delta G^{\dagger})^{-1}\big]_{t+\tau,t}\\ &= \frac{1}{2}\frac{1+c}{(1+\chi)^2}.\end{aligned} \tag{84}$$

This result is exact. For the volatility itself, which in view of (67)–(69) can be written as $\sigma^2 = \lim_{t\to\infty}\Xi(t,t)$, and which depends on the detailed short-time structure of the kernels G and D, one has to resort to approximations. One such approximation is motivated by a property of mean-field disordered systems with detailed balance, where fluctuation-dissipation theorems (FDT) allow one to gauge away the non-persistent parts of the response and correlation function while leaving averages containing a single time index and averages containing infinitely separated times unchanged (see e.g. [26]); for equilibrium systems the resulting simpler equations are exact. Although no general analogons of equilibrium FDTs are as yet known for non-equilibrium systems such as the MG, one could assume that the resulting recipe for removing non-persistent contributions to the various kernels also applies to non-equilibrium systems; this as yet ad hoc assumption has recently been applied with remarkable success to a similar non-equilibrium problem [25]. Here it would amount to removing all non-persistent parts of the various kernels, while retaining the relations $\delta\sum_t G_{tt'} = \chi$, $\frac{1}{\tau}\sum_{t\leqslant\tau} C_{t,t'} = c$ and $C_{tt} = 1$, i.e. one inserts

$$C_{t,t'} = c + (1-c)\delta_{tt'} \qquad G_{t,t'} = \chi\gamma(1-\gamma\delta)^{t-t'-1} \qquad (t > t') \tag{85}$$

and takes $\gamma\to 0$ at the end of the calculation. The only expressions containing G which will survive this limit, are those where each instance of G is summed over the whole history. More specifically, in the volatility

$$\sigma^2 = \tfrac{1}{2}\sum_{\ell\geqslant 0}\sum_{\ell'\geqslant 0}\big\langle[(GE)^{\ell}D(EG^{\dagger})^{\ell'}](0)\big\rangle_n \tag{86}$$

contractions like $\delta\sum_s G_{ts}G_{t's}$ cannot survive, as they are of order γ. Hence the Poisson average in (86) factorizes over the E matrices, and the only non-vanishing term involving the diagonal part of D is the term $\ell = \ell' = 0$. The result[3] is

$$\sigma^2 \approx \frac{1}{2}\frac{1+c}{(1+\chi)^2} + \frac{1}{2}(1-c). \tag{87}$$

This expression for the volatility depends only on persistent order parameters, and is independent of the learning rate $\tilde{\eta}$. Work is in progress to explore and understand the theoretical basis, if it exists, of (or disprove the correctness of, as the case may be) the elimination of the non-persistent parts in general mean-field disordered systems without detailed balance, which underlies the derivation of (87) from the exact expression (86) in the ergodic regime.

10. Relation between present exact solution and previous work

The present study improves upon and generalizes previous work on the on-line MG in several ways: it is exact for $N\to\infty$, it is a dynamical theory (with statics included as a spin-off), and it deals with a large family of decision noise definitions (including additive and multiplicative noise, and intermediates). In this section we compare the various approximations and assumptions made in previous studies with the exact solution, and also compare the solution of the on-line MG with that of the batch MG.

[3] Note: in the batch case [20, 21] one finds a slightly different expression.

10.1. Expressions for the diffusion matrix

The first study in which one finds an expression for the diffusion matrix of the microscopic on-line process is [15] (concerned with both additive and multiplicative noise). Comparison with the exact expressions (23), (24) shows that the matrix given in [15] can be regarded as an approximation obtained by disregarding fluctuations due to the selection of external information, and retaining only those generated by the decision noise. Note that, similar to (24), it contains the factors $(1 - \tanh^2[\beta q_i])$ and vanishes as $\beta \to \infty$. The diffusion term presented more recently (for additive noise) in [17] can also be seen as an approximation of (23), (24), obtained upon replacing $\sum_\mu \xi_k^\mu \xi_\ell^\mu \langle A^\mu[\boldsymbol{q}, \boldsymbol{z}]^2 \rangle_{\boldsymbol{z}}$ by $\lim_{\tau\to\infty} \tau^{-1} \sum_{t\leqslant\tau} \sum_\mu \xi_k^\mu \xi_\ell^\mu \frac{1}{p} \sum_\nu \langle A^\nu[\boldsymbol{q}(t), \boldsymbol{z}]^2 \rangle_{\boldsymbol{z}}$, i.e. by neglecting certain correlations and subsequently introducing an additional temporal average over $\boldsymbol{q}$.

Note that neither of the two studies [15, 17] attempt to solve the dynamics; the objective of [15] was to highlight the presence and role of the diffusion term in the microscopic equations (since at the time the microscopic laws were claimed to be effectively deterministic [11]). The authors of [17] have to restrict themselves to statics, since in their approximation and subsequent analysis they neglect the time dependence of the fluctuation term which they replace by the volatility (or, equivalently, they do not regard their volatility as time dependent). However, their approach does shed additional light on the nature of the phase transition at $\alpha = \alpha_c(T)$ and the system state for $\alpha < \alpha_c(T)$.

10.2. The impact of truncating the Kramers–Moyal expansion

Initially, in [11], the KM expansion was truncated after the Liouville term, leading to deterministic microscopic laws. In [15, 17] it was truncated after the Fokker–Planck term. The question we turn to now is: what is the effect of such truncations, assuming one would have taken the correct diffusion matrix (i.e. both terms (23) and (24))? This question is even more relevant in view of the fact that, for additive noise, the exact equations (75), (77), (78), which give the stationary solution in the ergodic regime, are identical to those found earlier in [11].

If we trace back our derivations, we find that the effect of truncating the KM expansion after the deterministic term or after the Fokker–Planck term is to replace the function ϕ in (46) by

$$\phi_{\mathrm{Det}}[\{x, w\}, u] = \mathrm{e}^{\delta \sum_t \mathrm{i} w_t (u + x_t)} \tag{88}$$

or by

$$\phi_{\mathrm{FP}}\,[\{x, w\}, u] = \mathrm{e}^{\delta \sum_t \mathrm{i} w_t (u + x_t) - \frac{1}{4}\tilde{\eta}\delta \sum_t w_t^2 (u + x_t)^2} \tag{89}$$

respectively. It turns out that both expressions can be written in the form (46), i.e.

$$\phi[\{x, w\}, u] = \left\langle \mathrm{e}^{\frac{1}{2}\mathrm{i}\tilde{\eta}\sum_t n_t w_t [u + x_t]} \right\rangle_n \tag{90}$$

$$\langle f[n_1, n_2, \ldots] \rangle_n = \sum_{n_1, n_2 \ldots \geqslant 0} \left[\prod_s P_{2\delta/\tilde{\eta}}[n_s]\; f[n_1, n_2, \ldots] \right] \tag{91}$$

but with alternative definitions for the statistics of the random variables $\{n_t\}$:

$$\text{Deterministic:} \quad P_a[n] = \delta[n - a] \tag{92}$$

$$\text{Fokker–Planck:} \quad P_a[n] = \frac{\mathrm{e}^{-\frac{1}{2}(n-a)^2/a}}{\sqrt{2\pi a}}. \tag{93}$$

Both truncations can thus be seen as approximations of the true (Poisson) distribution which describes the noise due to random information selection: in the deterministic case one replaces the true $P_{2\delta/\tilde{\eta}}[n]$ by a delta-distribution with the correct first moment $\langle n\rangle = 2\delta/\tilde{\eta}$, in the Fokker–Planck case one replaces the true $P_{2\delta/\tilde{\eta}}[n]$ by a Gaussian distribution and ensures that the first *two* moments $\langle n\rangle = 2\delta/\tilde{\eta}$ and $\langle n^2\rangle = (2\delta/\tilde{\eta})^2 + 2\delta/\tilde{\eta}$ are correct. The above representation allows us to continue with our original derivation, in spite of the truncations, right up to and including the equations describing the effective single-trader process (62)–(64), since the choice made for $P_a[n]$ only affects the kernels $R(t, t')$ and $\Sigma(t, t')$. The derivation of expression (65), however, only involved the first moment of $P_a[n]$ (see section 6), so *both* truncations would have led to the exact expression for $R(t, t')$. Surprisingly, the derivation of (66) only involved the first two moments of $P_a[n]$, so whereas the deterministic truncation would have led to an incorrect expression for the covariance matrix $\Sigma(t, t')$ (obtained by putting $\tilde{\eta} \to 0$ in (66)), the Fokker–Planck approximation would have led to the correct expression (66) and hence to the exact dynamical order parameter equations.

This explains why such truncations, although not a priori justified, here can lead to correct results. The deterministic theory would lead at most to correct equations in ergodic stationary states (where the learning rate of (66) drops out), but would fail to describe dynamical properties. The Fokker–Planck truncation would lead to the correct macroscopic dynamical theory, provided one uses the correct diffusion matrix, because the relevant order parameters of the present model fortunately turn out not to be sensitive to the (weak) non-Gaussian fluctuations.

10.3. Approximations of the volatility

Since the volatility (79) involves non-persistent order parameters (describing correlations over short temporal separations, even in the stationary state), it could not be calculated directly in any of the previous studies. Instead one had to resort to approximations.

We repeat an argument here which was first given in [10], and which applies in the ergodic $\alpha > \alpha_c(T)$ regime. Upon assuming that phase space is sampled ergodically by the process, one may replace the time and pattern presentation averages by an average over the equilibrium probability measure

$$\sigma^2 = \left[\left\langle \frac{1}{p}\sum_{\mu}\langle A^{\mu}[\boldsymbol{q}, \boldsymbol{z}]^2\rangle_{\boldsymbol{z}}\right\rangle\right]_{\mathrm{dis}}. \tag{94}$$

The decision noise average factorizes over sites, i.e. $\langle\sigma[q_i, z_i]\sigma[q_j, z_j]\rangle_{\boldsymbol{z}} = \langle\sigma[q_i, z_i]\rangle_{\boldsymbol{z}}\langle\sigma[q_j, z_j]\rangle_{\boldsymbol{z}}$ for $i \neq j$ (this is not an approximation). It was subsequently argued that in ergodic states the mean field character of the system means that also the ensemble average factorizes over sites, i.e. $\langle\sigma[q_i, z_i]\sigma[q_j, z_j]\rangle = \langle\sigma[q_i, z_i]\rangle\langle\sigma[q_j, z_j]\rangle$ for $i \neq j$. This should be regarded as an approximation, since, although for large N non-diagonal correlations can be assumed weak, they are also N in number and therefore cannot simply be discarded. The result of this mean field approximation is

$$\begin{aligned}\sigma^2 &\approx \frac{1}{p}\sum_{\mu}\left[\langle\langle A^{\mu}\rangle_{\boldsymbol{z}}\rangle^2\right]_{\mathrm{dis}} + \frac{1}{p}\sum_{\mu}\frac{1}{N}\sum_i\left[\xi_i^{\mu}\xi_i^{\mu}\left\{1 - \langle\langle\sigma[q_i, z]\rangle_z\rangle^2\right\}\right]_{\mathrm{dis}} \\ &= \frac{1}{p}\sum_{\mu}\left[\langle\langle A^{\mu}\rangle_{\boldsymbol{z}}\rangle^2\right]_{\mathrm{dis}} + \frac{1}{2}\left[1 - \frac{1}{N}\sum_i\langle\langle\sigma[q_i, z]\rangle_z\rangle^2\right]_{\mathrm{dis}} + \mathcal{O}(N^{-1/2})\end{aligned} \tag{95}$$

which then leads, in the ergodic regime and for additive decision noise, more or less directly to

$$\sigma^2 \approx \frac{1}{2}\frac{1+c}{(1+\chi)^2} + \frac{1}{2}(1-c) \tag{96}$$

which is equation (87). This identification shows that the exactness or otherwise of the (traditional) approximation (96) is crucially linked to the question of under which conditions non-persistent parts of dynamic order parameters can be 'transformed away' in ergodic non-equilibrium models (see section 9). It also emphasizes that such approximations must fail in the $\alpha < \alpha_c(T)$ regime, since in equilibrium systems such procedures are based on FDT and therefore typically reproduce the replica-symmetric solution.

Numerical evidence presented in [11, 12] shows that (96) is a very good approximation in the $\alpha > \alpha_c(T)$ region, until just above the transition point $\alpha_c(T)$. It is not clear whether the slight discrepancy just above $\alpha_c(T)$ is due to insufficient equilibration in the simulation, or because the approximation breaks down.

10.4. Relation with the dynamical solution of the batch MG

Finally, comparison shows that the present dynamical solution (62)–(64) for the on-line MG can be regarded as a straightforward continuous-time equivalent of the discrete-time dynamical solution of the batch MG [20, 21], obtained simply by substituting time derivatives for discrete differences and integral kernels for matrices, for any type of decision noise. The only real difference is in the occurrence of the learning rate $\tilde{\eta}$ in (66) (which makes sense, since it reflects fluctuations relating to the random choice of external information, which are absent by definition in the batch models), which will be responsible for differences between batch and on-line MG models in the transients and in the non-ergodic region. This learning rate term, however, does not affect the stationary state solution for $\alpha > \alpha_c(T)$, which explains why the stationary state of the batch models [20,21] and the locations $\alpha_c(T)$ of their ergodicity-breaking transitions were identical to those found earlier (by others) for the on-line MG.

11. Discussion

In this paper we have solved the dynamics of the on-line MG, with general types of decision noise (including additive and multiplicative decision noise as specific choices). We have done so by following the successful approach which also recently led to the exact solution of discrete-time batch versions of the MG [20, 21] (i.e. the application of generating functional techniques a la De Dominicis [19]), and we have dealt with the problems relating to temporal regularization, which occur only in on-line MGs, using the (exact) procedure of [18]. The end result is a macroscopic dynamical theory in the form of closed equations for correlation- and response functions (the dynamical order parameters of the problem) which are defined via an effective continuous-time single-trader process. These equations are exact for $N \to \infty$ (where N denotes the number traders), and they incorporate all static and dynamic properties of the on-line MG, both in the ergodic and in the non-ergodic regimes.

We have used our theory to resolve a number of open problems related to approximations and assumptions made in previous studies. For instance, we show why it is in principle not allowed to truncate the Kramers–Moyal expansion of the microscopic process after the Fokker–Planck term (let alone after the flow term), but why upon doing so one can for the present version of the MG still find the correct macroscopic equations, we confirm that the different diffusion matrices for the Fokker–Planck term in the process as proposed earlier by others are incomplete or approximate, and we indicate how previously proposed approximations involving the market volatility can be traced back to assumptions relating to ergodicity.

The macroscopic theory now available is not only exact, but also more comprehensive than its on-line predecessors: it deals with the full dynamics (with statics included as a by-product), and it generalizes the class of decision noise definitions. We also hope that our theory will end the discussions about which are the correct microscopic equations for the MG, and that it can be used in future as the canonical starting point for both analyses and generalizations (e.g. dynamics in the non-ergodic regime, or using the real market history as external information as originally proposed in [1]), and for approximations (for which there is still a need, since it is generally a non-trivial task to solve our macroscopic laws). This might well include some of the approximations which have already been proposed earlier, which can now be provided with transparent interpretations and with a guide for systematic improvement.

Acknowledgments

JAFH would like to thank the King's College London Association for support. ACCC and JAFH would like to thank the organizers of the 'International Seminar on Statistical Mechanics of Information Processing in Cooperative Systems' (Dresden, 2001) where many fruitful discussions helped shaping the present study.

References

[1] Challet D and Zhang Y-C 1997 *Physica* A **246** 407
[2] Arthur W B 1994 *Am. Econ. Assoc. Papers Proc.* **84** 406
[3] Challet D webpage http://www.unifr.ch/econophysics/minority
[4] Cavagna A 1999 *Phys. Rev.* E **59** R3783
[5] Savit R, Manuca R and Riolo R 1999 *Phys. Rev. Lett.* **82** 2203
[6] Cavagna A, Garrahan J P, Giardina I and Sherrington D 1999 *Phys. Rev. Lett.* **83** 4429
[7] Hart M, Jefferies P, Johnson N F and Hui P M 2001 *Physica* A **298** 537
[8] Hart M, Jefferies P and Johnson N F 2000 *Phys. Rev.* E **63** 017102
[9] Jefferies P, Hart M, Johnson N F and Hui P M 2000 *J. Phys. A: Math. Gen.* **33** L409
[10] Challet D and Marsili M 1999 *Phys. Rev.* E **60** 6271
[11] Challet D, Marsili M and Zecchina R 2000 *Phys. Rev. Lett.* **84** 1824
[12] Marsili M, Challet D and Zecchina R 2000 *Physica* A **280** 522
[13] Challet D, Marsili M and Zecchina R 2000 *Phys. Rev. Lett.* **85** 5008
[14] Cavagna A, Garrahan J P, Giardina I and Sherrington D 2000 *Phys. Rev. Lett.* **85** 5009
[15] Garrahan J P, Moro E and Sherrington D 2000 *Phys. Rev.* E **62** R9
[16] Marsili M and Challet D 2001 *Adv. Complex Syst.* **3** I
[17] Marsili M and Challet D 2001 *Preprint* cond-mat/0102257
[18] Bedeaux D, Lakatos-Lindenberg K and Shuler K 1971 *J. Math. Phys.* **12** 2116
[19] De Dominicis C 1978 *Phys. Rev.* B **18** 4913
[20] Heimel J A F and Coolen A C C 2001 *Phys. Rev.* E **63** 056121
[21] Coolen A C C, Heimel J A F and Sherrington D 2001 *Preprint* cond-mat/0106635 (*Phys. Rev.* E, at press)
[22] Cugliandolo L F and Kurchan J 1995 *Phil. Mag.* B **71** 501
[23] De Martino A and Marsili M 2000 *J. Phys. A: Math. Gen.* **34** 2525–37
[24] Heimel J A F and De Martino A 2001 *J. Phys. A: Math. Gen.* **34** L539
[25] Heimel J A F and Coolen A C C 2001 *J. Phys. A: Math. Gen.* **34** 9009 (cond-mat/0102272)
[26] Fisher K H and Hertz J A 1991 *Spin Glasses* (Cambridge: Cambridge University Press)